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Transcendental Inverse Eigenvalue Problem Associated with Longitudinal Vibrations in Rods

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The dynamics of a continuous system is represented by a transcendental eigenvalue problem, whereas the associated discrete approximating model is characterized by an algebraic eigenvalue problem. The fact that the asymptotic behavior of the eigenvalues of a transcendental eigenvalue problem is different from the asymptotic behavior for an approximating algebraic eigenvalue problem forms an obstacle to solving the inverse problem of reconstructing the physical parameters of a continuous system based on a discrete model. To overcome this obstacle, a new mathematical model is presented where the transcendental eigenvalue problem associated with a continuous system with varying physical properties is approximated by a continuous system with piecewise constant physical parameters. An algorithm for solving the associated inverse eigenvalue problem is presented. Numerical examples demonstrate that the physical properties of a continuous system can be reconstructed by using this approach.

Nomenclature

A, B	=	matrices representing transcendental eigenvalue problems
A_i, B_i	=	arbitrary constants in general solutions of ordinary differential equations
c_i	=	speed of sound in the ith portion of the rod
F_i	=	the ith equation defining the inverse problem
h	=	length of the element in the finite difference model
i	=	index (integer)
J	=	Jacobian matrix of partial derivatives
j	=	index (integer)
K	=	stiffness matrix
k	=	element stiffness
M	=	mass matrix
m	=	element mass
m_T	=	total mass of the rod
n	=	model order
P	=	a typical $n \times n$ matrix
$\hat{P}$	=	$(n-1) \times (n-1)$ leading principal submatrix of P
p_i	=	axial rigidity in the ith portion of the rod
r_i	=	normalized rigidity in the ith portion of the rod
t	=	time variable
u_i	=	axial displacement in the ith portion of the rod
v_i	=	eigenfunction associated with the ith portion of the rod
$\boldsymbol{x}$	=	vector of unknown parameters
x	=	position variable
$\boldsymbol{z}$	=	eigenvector of a transcendental matrix
$\boldsymbol{z}_A$	=	eigenvector of A
$\boldsymbol{z}_B$	=	eigenvector of B
$\boldsymbol{\delta}$	=	vector of correction to the unknown parameters
δ_i	=	correction to the ith unknown parameter
ε	=	error tolerance
η_i	=	the ith lowest eigenvalue of the fixed-fixed rod
λ	=	eigenvalue of a transcendental matrix
λ_i	=	ith lowest eigenvalue of the fixed–free rod
μ	=	natural frequencies of the fixed–fixed rod
ρ_i	=	mass density in the ith portion of the rod
ϕ	=	right eigenvectors of P
ψ	=	left eigenvectors of P
ω	=	natural frequencies of the fixed–free rod
$\mathbf{0}$	=	zero vector

I. Introduction

By the transcendental eigenvalue problem we refer to the problem of finding the nontrivial solutions $\lambda, \boldsymbol{z} \neq \mathbf{0}$, of

$$A(\lambda)\boldsymbol{z} = \mathbf{0} \tag{1}$$

where the elements of A are transcendental functions in λ. Such problems arise naturally in the area of structural stability and vibrations of continuous systems. Consider, for example, the piecewise uniform rod shown in Fig. 1a. The longitudinal oscillations of the rod are governed by the partial differential equations

$$\frac{\partial}{\partial x}\left(p_1 \frac{\partial u_1}{\partial x}\right) = \rho_1 \frac{\partial^2 u_1}{\partial t^2}, \qquad 0 < x < 0.5, \qquad t > 0 \tag{2}$$

$$\frac{\partial}{\partial x}\left(p_2 \frac{\partial u_2}{\partial x}\right) = \rho_2 \frac{\partial^2 u_2}{\partial t^2}, \qquad 0.5 < x < 1, \qquad t > 0 \tag{3}$$

the matching displacement and force at $x = 0.5$,

$$u_1(0.5, t) = u_2(0.5, t) \tag{4}$$

$$p_1 \frac{\partial u_1}{\partial x}\bigg|_{x=0.5,t} = p_2 \frac{\partial u_2}{\partial x}\bigg|_{x=0.5,t} \tag{5}$$

and the boundary conditions

$$u_1(0, t) = \frac{\partial u_2}{\partial x}\bigg|_{x=1,t} = 0 \tag{6}$$

Noting that p_1 and p_2 are constants, separation of variables $u_i(x, t) = v_i(x)\sin\omega t$, $i = 1, 2$, applied to Eqs. (2–6) leads to the Sturm–Liouville eigenvalue problem

$$p_1 v_1'' + \lambda\rho_1 v_1 = 0, \qquad 0 < x < 0.5 \tag{7}$$

$$p_2 v_2'' + \lambda\rho_2 v_2 = 0, \qquad 0.5 < x < 1 \tag{8}$$

$$v_1(0.5) = v_2(0.5) \tag{9}$$

$$p_1 v_1'(0.5) = p_2 v_2'(0.5) \tag{10}$$

$$v_1(0) = v_2'(1) = 0 \tag{11}$$

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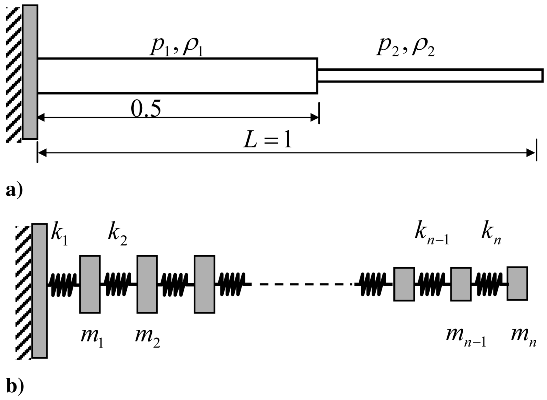

Fig. 1 Axially vibrating rod: a) piecewise fixed–free rod divided into two equal portions and b) approximated lumped mass–spring model of order *n* for the rod.

where $\lambda \equiv \omega^2$, and primes denote derivatives with respect to x. The general solution of Eqs. (7) and (8) is given by

$$v_i(x) = A_i \sin(\omega x/c_i) + B_i \cos(\omega x/c_i), \qquad i = 1, 2 \tag{12}$$

where $c_i = \sqrt{(p_i/\rho_i)}, i = 1, 2$. Substituting Eq. (12) into Eqs. (9–11) yields the transcendental eigenvalue problem

$$\begin{bmatrix} \sin(\omega/2c_1) & \cos(\omega/2c_1) & -\sin(\omega/2c_2) & -\cos(\omega/2c_2) \\ p_1c_2\cos(\omega/2c_1) & -p_1c_2\sin(\omega/2c_1) & -p_2c_1\cos(\omega/2c_2) & p_2c_1\sin(\omega/2c_2) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\omega/c_2) & -\sin(\omega/c_2) \end{bmatrix} \begin{pmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{13}$$

that is, $A(\omega)z = \mathbf{0}$, with the obvious definition of $A(\omega)$ and z. The direct eigenvalue problem consists of finding the natural frequencies ω_i that make $A(\omega_i)$ singular and their corresponding eigenvectors $z_i \neq \mathbf{0}$, $i = 1, 2, \ldots,$ knowing the physical parameters p_j and c_j, $j = 1, 2$. Once ω_i and z_i are found, a mode of the system can be determined via Eq. (12). We have described in Ref. 1 an effective numerical procedure for solving this problem. The present paper deals with the inverse eigenvalue problem of determining the physical parameters of the system knowing some of its natural frequencies.

Frequently, instead of solving the transcendental eigenvalue problem (13), the continuous system (2–6) is transformed by finite difference, finite element, or lumped parameter methods to a discrete form. For example, dividing the rod of Fig. 1a into even equally spaced elements of length $h = 1/n$, lumping the mass of the various element to point masses,

$$m_i = \begin{cases} h\rho_1, & i = 1, 2, \ldots, n/2 \\ h\rho_2, & i = n/2 + 1, n/2 + 2, \ldots, n \end{cases} \tag{14}$$

and expressing the rigidity between two consecutive mass by a linear spring of constant

$$k_i = \begin{cases} p_1/h, & i = 1, 2, \ldots, n/2 \\ p_2/h, & i = n/2 + 1, n/2 + 2, \ldots, n \end{cases} \tag{15}$$

we obtain the algebraic eigenvalue problem corresponding to the discrete model of the rod, which has the form

$$(K - \lambda M)v = \mathbf{0} \tag{16}$$

where

$$M = \mathrm{diag}\{m_1 \quad m_2 \quad \cdots \quad m_n\}^T \tag{17}$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & & & \\ -k_2 & k_2 + k_3 & -k_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -k_{n-1} & k_{n-1} + k_n & -k_n \\ & & & -k_n & k_n \end{bmatrix} \tag{18}$$

and where v is the eigenvector corresponding to the eigenvalue λ.

Research in the area of inverse eigenvalue problems associated with vibratory systems is summarized in a monograph[2] and review papers.[3–6] It is well known that in theory a single set of eigenvalues λ_i, $i = 1, 2, \ldots,$ cannot determine the distributions of the physical parameters $p(x)$ and $\rho(x)$ of a rod uniquely. These parameters can be determined if a second set of eigenvalues, corresponding to the fixed–fixed boundary conditions η_i, $i = 1, 2, \ldots,$ is also given. The spectrum used for such a determination must satisfy the interlacing property

$$0 < \lambda_1 < \eta_1 < \lambda_2 < \eta_2 < \lambda_3 < \eta_3 \cdots \tag{19}$$

to ensure that the solution is physical, that is, one with positive masses and springs. The uniqueness of the reconstructed parameters is determined by adding a nonspectral parameter such as the total mass. In theory, the reconstruction process requires knowing the complete sets of eigenvalues associated with the continuous system. However, acquiring a large number of eigenvalues is not possible in practice. Inverse eigenvalue problems associated with discrete vibratory systems have been studied also in Refs. 7 and 8. It has been shown in Ref. 8 that for a given set of distinct eigenvalues λ_i, $i = 1, 2, \ldots, n$, η_i, $i = 1, 2, \ldots, n-1$, satisfying relation (19), one can reconstruct a symmetric tridiagonal matrix $P \in R^{n \times n}$ such that the eigenvalues of P are λ_i, $i = 1, 2, \ldots, n$, and the eigenvalues of its leading principal submatrix $\hat{P} \in R^{n-1 \times n-1}$ are η_i, $i = 1, 2, \ldots, n-1$.

In this paper we consider the problem of reconstructing the physical parameters of nonuniform axially vibrating rods by solving certain transcendental eigenvalue problems associated with the continuous system. We begin with demonstrating the main obstacle in solving such inverse problems by reconstructing a discrete mass–spring system based on the spectrum of the continuous system. This illustration demonstrates the motivation of addressing the existing open problem of reconstructing the continuous system accurately by using a finite number of eigenvalues. In Sec. III we formulate the reconstruction problem for a piecewise uniform continuous rod model as a transcendental eigenvalue problem. We then develop a numerical method for solving the associated inverse problem of reconstructing the variable physical parameters of the rod along its length in Sec. IV. Illustrative examples are given in Sec. V, and conclusions are drawn in Sec. VI.

II. Motivation

To demonstrate the current state of the art associated with reconstructing the physical parameters of a continuous system based on a discrete model, we consider the example of the piecewise rod shown in Fig. 1a for the special case where $p_1 = p_2 = \rho_1 = \rho_2 = 1$, that is, a uniform rod with unit physical properties. The eigenvalues of such a fixed–free uniform rod are

$$\lambda_j = (2j-1)^2\pi^2/4, \qquad j = 1, 2, 3, \ldots \tag{20}$$

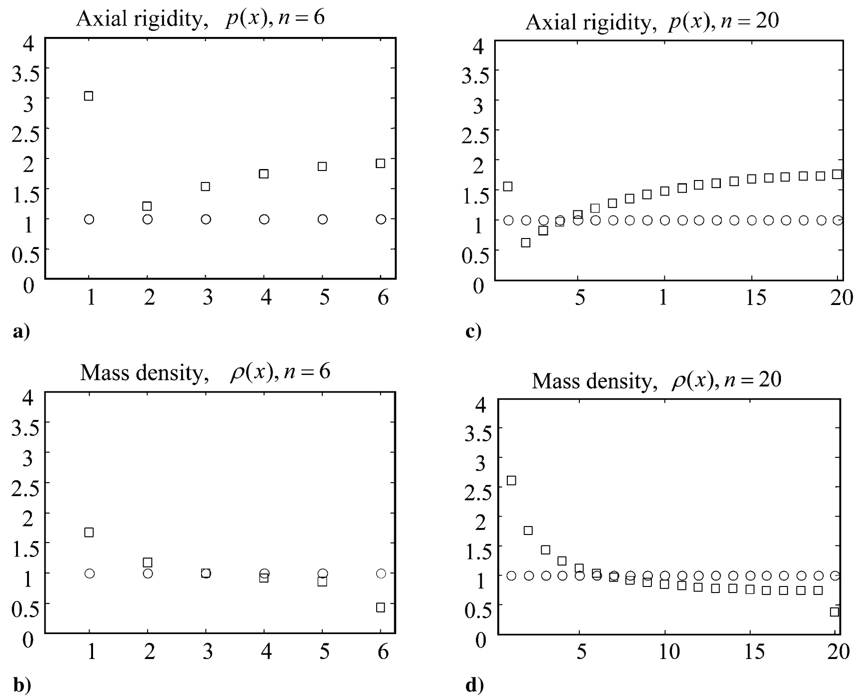

Fig. 2 Parameter estimation based on lumped parameter model: ○, physical parameters of the continuous uniform rod and □, physical parameters of the uniform rod identified by the reconstruction method.

If this uniform rod is fixed at both its ends, then its eigenvalues are

$$\eta_j = j^2\pi^2, \qquad j = 1, 2, 3, \ldots \tag{21}$$

We try to determine the physical parameters of this rod using the classical method of reconstructing a discrete model of dimension n from λ_i, $i = 1, 2, \ldots, n$, η_j, $j = 1, 2, \ldots, n-1$, and the total mass of the rod m_T, by the method described for example in Ref. 2. For model order $n = 6$ the discrete parameters p_i and ρ_i shown in Figs. 2a and 2b are obtained. They clearly deviate significantly from the expected constant values associated with the uniform rod. We note that the global estimation of the physical parameters is not much improved when the model order is increased to $n = 20$, as demonstrated in Figs. 2c and 2d. Although the eigenvalues of the discrete model do converge to those of the continuous system as the model order n is increased, the higher eigenvalues converge more slowly than the lower eigenvalues. Therefore, for any given value n, the results of the discrete inverse eigenvalue problem are skewed due to errors in the higher eigenvalues. This phenomenon has been attributed in Ref. 9 to the different asymptotic behavior of the spectra of the continuous system and its associated discrete model. In other words, the characteristic polynomial associated with the algebraic eigenvalue problem lacks the ability to approximate the transcendental function associated with the eigenvalues of the continuous model accurately. Hence the inaccurate eigenvalues associated with the discrete model deteriorate the reconstruction process regardless of the model order. Boley and Golub have summarized this phenomenon in Ref. 10 by stating, "There are many similarities between the matrix problems and the continuous problems, but numerical evidence demonstrates that the solution to the matrix problem is not a good approximation to the continuous one."

The motivation of this paper is to develop a new low-order mathematical model that can be used for estimating the physical parameters of the continuous system using a small number of eigenvalues. The main idea here is to approximate a nonuniform continuous system by another continuous system with piecewise constant physical parameters. The mathematical modeling and the formulation of the approximated low-order piecewise continuous system leads to a transcendental eigenvalue problem of the form given by Eq. (13). We have used this mathematical model in Ref. 11 in reconstructing the physical parameters of the continuous system from its eigenvalues. The formulation of this problem is now developed in Sec. III.

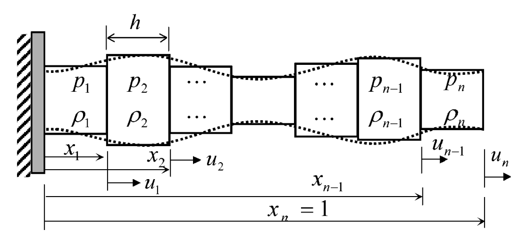

Fig. 3 Nonuniform axially vibrating rod approximated by *n* piecewise constant rods with constant physical parameters.

III. Transcendental Eigenvalue Model

Consider a nonuniform axially vibrating rod, shown in Fig. 3, that is approximated by another rod of model order n with piecewise constant physical properties. The eigenvalue problem for this piecewise constant rod can be expressed as

$$p_i v_i'' + \lambda\rho_i v_i = 0, \qquad x_{i-1} < x < x_i \quad \text{for} \quad i = 1, 2, 3, \ldots, n \tag{22}$$

where p_i and ρ_i are constants, and $x_0 = 0$ and $x_n = L$ are the left and right boundary points. The continuity in displacement and force between the two adjacent piecewise elements is given by the following matching conditions:

$$v_i(x_i) = v_{i+1}(x_i), \qquad p_i v_i'(x_i) = p_{i+1} v_{i+1}'(x_i)$$
$$i = 1, 2, 3, \ldots, n-1 \tag{23}$$

The fixed-free configuration of the rod leads to the boundary condition

$$v_1(x_0) = v_n'(x_n) = 0 \tag{24}$$

Similarly,

$$v_1(x_0) = v_n(x_n) = 0 \tag{25}$$

are the boundary conditions associated with the fixed–fixed configuration. By considering the general solution of the form

$$v_i(x) = A_i \sin(\omega x / c_i) + B_i \cos(\omega x / c_i), \qquad i = 1, 2, 3, \ldots, n \tag{26}$$

and by applying the matching and boundary conditions, we obtain the transcendental eigenvalue problem of the form $A(\omega)z = \mathbf{0}$.

For example, the eigenvalue problem (13) corresponds to the fixed–free rod that is shown in Fig. 1a. This is a special case of the piecewise nonuniform rod where $n = 2$. For this problem, Eq. (26) is reduced to that described in Eq. (13) for the fixed–free rod, and to

$$\begin{bmatrix} \sin(\mu/2c_1) & \cos(\mu/2c_1) & -\sin(\mu/2c_2) & -\cos(\mu/2c_2) \\ p_1c_2\cos(\mu/2c_1) & -p_1c_2\sin(\mu/2c_1) & -p_2c_1\cos(\mu/2c_2) & p_2c_1\sin(\mu/2c_2) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sin(\mu/c_2) & \cos(\mu/c_2) \end{bmatrix} \begin{pmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{27}$$

for the fixed–fixed configuration, where $\eta = \mu^2$ is a eigenvalue of the fixed–fixed rod. Hence, the nontrivial solutions of the transcendental eigenvalue problems (13) and (27) furnish two sets of spectra ω_i and μ_i, $i = 1, 2, 3, \ldots,$ corresponding to the eigenvalues of the rod with two different boundary conditions. The problem considered in the paper is to estimate the physical parameters $p_1, p_2, \ldots, p_n$ and $\rho_1, \rho_2, \ldots, \rho_n$ of the various sections of the rod using spectral data. More specifically, we denote the ratios $r_i = p_i/p_1$, $i = 1, 2, \ldots, n$, and define the following problem.

Given: The eigenvalues $\omega_1, \omega_2, \ldots, \omega_n$ and $\mu_1, \mu_2, \ldots, \mu_{n-1}$ of the rod in its fixed–free and fixed–fixed configurations.

Find: The physical parameters $c_1, c_2, \ldots, c_n$ and $r_2, r_3, \ldots, r_n$.

For example, in the case of the piecewise rod of Fig. 1a the characteristic equations associated with the fixed–free and fixed–fixed configurations of the rod can be expressed as $\det[A(\omega_i)] = 0$, $i = 1, 2$, and $\det[B(\mu_1)] = 0$, respectively, where

$$A(\omega_i) = \begin{bmatrix} \sin(\omega_i/2c_1) & \cos(\omega_i/2c_1) & -\sin(\omega_i/2c_2) & -\cos(\omega_i/2c_2) \\ c_2\cos(\omega_i/2c_1) & -c_2\sin(\omega_i/2c_1) & -r_2c_1\cos(\omega_i/2c_2) & r_2c_1\sin(\omega_i/2c_2) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\omega_i/c_2) & -\sin(\omega_i/c_2) \end{bmatrix} \tag{28}$$

$$B(\mu_1) = \begin{bmatrix} \sin(\mu_1/2c_1) & \cos(\mu_1/2c_1) & -\sin(\mu_1/2c_2) & -\cos(\mu_1/2c_2) \\ c_2\cos(\mu_1/2c_1) & -c_2\sin(\mu_1/2c_1) & -r_2c_1\cos(\mu_1/2c_2) & r_2c_1\sin(\mu_1/2c_2) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sin(\mu_1/c_2) & \cos(\mu_1/c_2) \end{bmatrix} \tag{29}$$

Our aim is to find the three parameters c_1, c_2, and r_2 that render $\det[A(\omega_i)] = 0$ for ω_1, ω_2, and $\det[B(\mu_1)] = 0$.

For this problem we denote the three equations associated with (28) and (29) as

$$F_i(\boldsymbol{x}) = \begin{cases} \det[A(\omega_i)] = 0, & i = 1, 2 \\ \det[B(\mu_1)] = 0, & i = 3 \end{cases} \tag{30}$$

where $\boldsymbol{x}$ is the vector of unknown variables $\boldsymbol{x} = (c_1 \;\; c_2 \;\; r_2)^T$. The system of equations (30) can then be solved for the three unknowns using Newton's method. In the next section we will show how these equations for $n \geq 2$ can be solved efficiently without expanding the determinants explicitly and without performing symbolic manipulations.

IV. Algorithm for Solving the Inverse Problem

The system of equations (30) corresponding to the case where $n \geq 2$ is now considered. If the rod is composed of n piecewise constant portions then we can define $2n \times 2n$ matrices $A(\omega)$ and $B(\mu)$ such that the system of equations

$$A(\omega)z_A = \mathbf{0} \tag{31}$$

$$B(\mu)z_B = \mathbf{0} \tag{32}$$

represents the matching and boundary conditions of the rod under the fixed–free and fixed–fixed configurations, respectively. We then denote

$$F_i(\boldsymbol{x}) = \begin{cases} \det[A(\omega_i)] = 0, & i = 1, 2, \ldots, n \\ \det[B(\mu_{i-n})] = 0, & i = n+1, n+2, \ldots, 2n-1 \end{cases} \tag{33}$$

where

$$\boldsymbol{x} = (x_1 \;\; x_2 \;\; \cdots \;\; x_{2n-1})^T = (c_1 \;\; \cdots \;\; c_n \;\; r_2 \;\; \cdots \;\; r_n)^T \tag{34}$$

is the vector of unknown parameters. We further define

$$F(\boldsymbol{x}) = [F_1(\boldsymbol{x}) \;\; F_2(\boldsymbol{x}) \;\; \cdots \;\; F_{2n-1}(\boldsymbol{x})]^T \tag{35}$$

We denote an approximate initial guess of the vector of unknown parameters by $\boldsymbol{x}^{(0)}$, and more generally $\boldsymbol{x}^{(k)}$ is the approximate value of $\boldsymbol{x}$ during the kth iterative step. Then Taylor series expansion of F in the neighborhood of $\boldsymbol{x}^{(k)}$ gives

$$F\left(\boldsymbol{x}^{(k+1)}\right) = F\left(\boldsymbol{x}^{(k)} + \boldsymbol{\delta}^{(k)}\right) = F\left(\boldsymbol{x}^{(k)}\right) + J^{(k)}\boldsymbol{\delta}^{(k)} + O\left(\boldsymbol{\delta}^{(k)}\right)^2 = \mathbf{0} \tag{36}$$

where the Jacobian matrix $J \in R^{(2n-1)\times(2n-1)}$ is defined as

$$J = \begin{bmatrix} \partial F_1/\partial x_1 & \partial F_1/\partial x_2 & \cdots & \partial F_1/\partial x_{2n-1} \\ \partial F_2/\partial x_1 & \partial F_2/\partial x_2 & \cdots & \partial F_2/\partial x_{2n-1} \\ \vdots & \vdots & \vdots & \vdots \\ \partial F_{2n-1}/\partial x_1 & \partial F_{2n-1}/\partial x_2 & \cdots & \partial F_{2n-1}/\partial x_{2n-1} \end{bmatrix} \tag{37}$$

$$\boldsymbol{\delta}^{(k)} = \left(\delta_1^{(k)} \;\; \delta_2^{(k)} \;\; \cdots \;\; \delta_n^{(k)}\right)^T \tag{38}$$

is the vector of correction to $\boldsymbol{x}^{(k)}$ during the kth iteration. Neglecting second- and higher-order terms in Eq. (36), the vector $\boldsymbol{\delta}^{(k)}$ is determined by solving the set of linear equations

$$J^{(k)}\boldsymbol{\delta}^{(k)} = -F\left(\boldsymbol{x}^{(k)}\right) \tag{39}$$

and the improved estimate is thus obtained by

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} + \boldsymbol{\delta}^{(k)} \tag{40}$$

by virtue of Eq. (36). Hence, starting from initial guess $\boldsymbol{x}^{(0)}$, the system (39) can solved successively for $k = 0, 1, \ldots,$ and the estimation can be modified according to Eq. (40) until the correction vector $\boldsymbol{\delta}^{(k)}$ is negligible in the sense that $\|\boldsymbol{\delta}^{(k)}\| < \varepsilon$ for a given small positive tolerance of error ε.

We now develop a practical numerical method for evaluating the Jacobian matrix (37). We note that the partial derivatives $\partial F_i/\partial x_j$ may be obtained by an explicit expansion of the determinant only for cases where n is small, because the computational complexity of expanding a determinant of a $2n \times 2n$ matrix is on the order of $2n!$ flops. To overcome this difficulty we invoke the definition of the determinant of a matrix as the product of its eigenvalues.[12] Hence

$$F_i\left(\boldsymbol{x}^{(k)}\right) = \begin{cases} \prod_{j=1}^{2n} \lambda_j[A(\omega_i)], & i = 1, 2, \ldots, n \\ \prod_{j=1}^{2n} \lambda_j[B(\mu_{i-n})], & i = n+1, n+2, \ldots, 2n-1 \end{cases} \tag{41}$$

where $\lambda_i(P)$ is the ith eigenvalue of the matrix P. Hence, by the chain rule,

$$\frac{\partial F_i}{\partial x_j} = \begin{cases} \sum_{r=1}^{2n} \frac{\partial \lambda_r(A(\omega_i))}{\partial x_j} \prod_{\substack{s=1 \\ s \neq j}}^{2n} \lambda_s[A(\omega_i)], & i = 1, 2, \ldots, n \\ \sum_{r=1}^{2n} \frac{\partial \lambda_r[B(\mu_{i-n})]}{\partial x_j} \prod_{\substack{s=1 \\ s \neq j}}^{2n} \lambda_s[B(\mu_{i-n})], & i = n+1, \quad n+2, \ldots, 2n-1 \end{cases} \tag{42}$$

However, for an arbitrary matrix $P(\xi)$ we have

$$\frac{\partial \lambda(P)}{\partial \xi} = \frac{\psi^T(\partial P/\partial \xi)\varphi}{\psi^T \varphi} \tag{43}$$

Hence, the partial derivatives in Eq. (42) can be calculated via Eq. (43) by evaluating the left and right eigenpairs of $A(\omega_i)$ and $B(\mu_i)$. The Jacobian matrix J is then determined through Eq. (42) by using standard eigenpairs extraction routines.

V. Examples

We now demonstrate how the method developed can be applied in practice by considering the following examples.

Example 5.1: We wish to reconstruct the physical parameters of the unit length piecewise continuous rod shown in Fig. 1a. The given spectral data are

$$\omega_1 = 2\cos^{-1}\left(\sqrt{3}/3\right), \qquad \omega_2 = 2\pi - 2\cos^{-1}\left(\sqrt{3}/3\right)$$
$$\mu_1 = \pi \tag{44}$$

We use an initial guess,

$$\boldsymbol{x}^{(0)} = \left(c_1^{(0)} \quad c_2^{(0)} \quad r_2^{(0)}\right)^T = (0.9 \quad 0.9 \quad 0.4)^T \tag{45}$$

and obtain after five iterations using the algorithm of Sec. IV

$$\boldsymbol{x}^{(p)} = (1 \quad 1 \quad 0.5)^T \tag{46}$$

The reconstructed rod has rigidity

$$p(x) = \begin{cases} \alpha & 0 < x < 0.5 \\ 0.5\alpha & 0.5 < x < 1 \end{cases} \tag{47}$$

and density

$$\rho(x) = \begin{cases} \alpha & 0 < x < 0.5 \\ 0.5\alpha & 0.5 < x < 1 \end{cases} \tag{48}$$

where α is an arbitrary constant. It can be confirmed that the first two eigenvalues of the system described by Eqs. (7–11) with $p_1 = \rho_1 = \alpha$ and $p_2 = \rho_2 = 0.5\alpha$ are, as required,

$$\omega_1 = 2\cos^{-1}\left(\sqrt{3}/3\right), \qquad \omega_2 = 2\pi - 2\cos^{-1}\left(\sqrt{3}/3\right) \tag{49}$$

With the same rigidity and density functions the eigenvalue problem defined by Eqs. (7–10) and $v_1(0) = v_2(1) = 0$ has lowest eigenvalue $\mu_1 = \pi$.

In the course of simulating the algorithm developed we found that there are in fact other rods that have the same two eigenvalues $2\cos^{-1}(\sqrt{3}/3)$ and $2\pi - 2\cos^{-1}(\sqrt{3}/3)$ associated with the fixed–free configuration and the eigenvalue π for the fixed–fixed configuration. They do not correspond, however, to the two lowest eigenvalues ω_1, ω_2 of the fixed–free rod and to the lowest eigenvalue μ_1 of the fixed–fixed rod, as required. Table 1 shows other rods that have such eigenvalues, together with their corresponding indices.

Example 5.2: Consider a fixed–free rod with cross-sectional area $A(x) = e^x$ and with unit rigidity, density, and length. The eigenvalue problem associated with this rod is

$$\begin{cases} (e^x v')' + \lambda e^x v = 0, & 0 < x < 1 \\ v(0) = 0, & v'(1) = 0 \end{cases} \tag{50}$$

The eigenvalues of the rod are the roots of (see, e.g., Ref. 13)

$$\tan\left(\sqrt{\lambda - \tfrac{1}{4}}\right) - 2\left(\sqrt{\lambda - \tfrac{1}{4}}\right) = 0 \tag{51}$$

If the exponential rod is fixed at both ends then its eigenvalues are the roots of

$$\sin\left(\sqrt{\mu - \tfrac{1}{4}}\right) = 0 \tag{52}$$

The first eight eigenvalues of the rod under the fixed–free and fixed–fixed configurations are shown in Table 2.

We have used the first eight eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_8$ of the fixed–free rod, the first seven eigenvalues $\mu_1, \mu_2, \ldots, \mu_7$ of the fixed–fixed rod, and the total mass $m = \exp(1) - 1$ to reconstruct the physical parameters of the rod. From these values the cross-sectional area of the rod is obtained as shown in Fig. 4. The identification looks quite satisfactory.

Table 1 Rods with the required eigenvalues

c_1	c_2	r_2	$2\cos^{-1}(\sqrt{3}/3)$	$2\pi - 2\cos^{-1}(\sqrt{3}/3)$	π
1	1	1/2	ω_1	ω_2	μ_1
1/2	1/2	1/8	ω_2	ω_3	μ_2
1/3	1/5	3/22	ω_3	ω_6	μ_4
1/5	1/5	1/242	ω_4	ω_7	μ_5
1/8	1/4	17/32	ω_4	ω_9	μ_6
1/6	1/6	69/40	ω_3	ω_6	μ_4

Table 2 Eigenvalues of the rod with exponential cross-sectional area

i	λ_i	μ_i
1	1.608533	10.119604
2	21.448812	39.728418
3	60.932289	89.076440
4	120.151266	158.163670
5	199.108652	246.990110
6	297.804973	355.555758
7	416.240385	483.860616
8	554.414947	631.904682

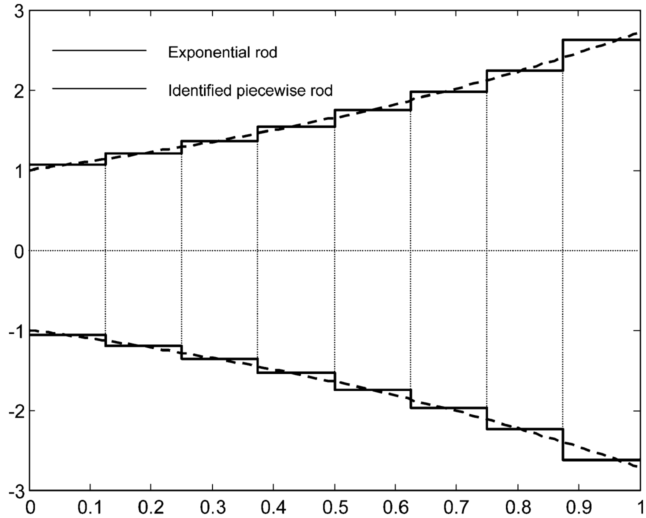

Fig. 4 Rod with exponential cross-sectional area and its reconstructed piecewise continuous rod model.

VI. Conclusions

It is well known that due to the asymptotic dissimilarity of the spectrum of a continuous system and that associated with its discrete model, it is not possible to reconstruct the physical parameters of a continuous axially vibrating rod accurately by using the classic inverse eigenvalue problem of reconstructing a mass–spring chain from spectral data. To overcome this difficulty we have proposed the reconstruction of a piecewise uniform continuous rod model by using the eigenvalues and the constrained eigenvalues of the nonuniform continuous rod. It should be noted that these spectral sets are the poles and zeros of the collocated frequency response function corresponding to the free end of the rod. It is well known that a necessary and sufficient condition for having such a physical rod, with positive cross-sectional area throughout, is that the interlacing property (19) holds. In practice, because the spectrum is obtained experimentally by modal tests, the spectral data should satisfy property (19) naturally. This procedure leads to a transcendental inverse eigenvalue problem, which has been formulated. An algorithm for solving the problem has been described. Examples of small-dimensional problems show that with the newly developed method the physical parameters of a nonuniform axially vibrating continuous rod can be estimated from small numbers of eigenvalues with reasonable accuracy.

With slight modifications, the method presented could be extended straightforwardly to solve the inverse problem of the transversely vibrating beam. For this case the physical parameters are determined by three spectral sets associated with three boundary conditions. It appears, however, that the method cannot be extended in a simple manner to two-dimensional structures such as the vibrating plate because the basic theory of inverse problems for two-dimensional structures is not yet fully developed.

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